Testing Normality Against the Logistic Distribution Using Saddlepoint Approximation

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Abstract

We consider the problem of testing normality against the logistic distribution, based on a random sample of observations. Since the two families are separate (non nested), the ratio of maximized likelihoods (RML) statistic does not have the usual asymptotic chi-square distribution. We derive the saddlepoint approximation to the distribution of the RML statistic and show that this approximation is more accurate than the normal and Edgeworth approximations, especially for tail probabilities that are the main values of interest in hypothesis testing. It is also shown that this test is almost identical to the most powerful invariant test.

Keywords Edgeworth expansion; Likelihood ratio test; Most powerful invariant test; Ratio of maximized likelihoods (RML); Tail probability approximation.

Subject Classification 62F03; 62F05.

1. Introduction

Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with unknown density function $h(x)$, and consider the problem of testing normality against the logistic distribution:

$$H_0 : h(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{vs.} \quad H_1 : h(x) = \frac{e^{-(x-a)/b}}{b(1+e^{-(x-a)/b})^2} \tag{1.1}$$

(the roles of H_0 and H_1 may be reversed). Since the two families of distributions in H_0 and H_1 are *separate* families, the ratio of maximized likelihoods (RML) statistic does not have the usual asymptotic chi-square distribution (Cox, 1961). In this article, we apply saddlepoint techniques, as developed in Rasekhi and Sadooghi-Alvandi (2008), to approximate the distribution of the RML statistic. In general, saddlepoint approximations are more accurate than normal approximations and Edgeworth approximations, especially for tail probabilities (which are the values of main interest in hypothesis testing problems). This is confirmed by our simulations,

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which show that the saddlepoint approximation is satisfactory even for small sample sizes.

Note that in a hypothesis-testing formulation, the two families are not treated symmetrically. We therefore also consider the case of testing

$$H_0 : h(x) = \frac{e^{-(x-a)/b}}{b(1+e^{-(x-a)/b})^2} \quad \text{vs.} \quad H_1 : h(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \tag{1.2}$$

and derive the saddlepoint approximation to the distribution of the RML statistic for this case also. (As noted in Rasekhi and Sadooghi-Alvandi, 2008, our results are also useful when the two families are treated symmetrically, i.e., when the problem is that of model selection or discrimination—rather than hypothesis testing.)

Note also that there is a close similarity between the normal and logistic distributions. In fact, this similarity has been used to derive approximations for the normal distribution function based on that of the logistic distribution, see, e.g., Chew (1968) and Lin (1990). Also, as noted by Johnson et al. (1995, p. 119), sometimes the normal distribution is replaced by the logistic distribution to simplify analysis. However, they also note that "Such substitution must be done with care and understanding of the similarities between the two distributions" ... "although there is a close similarity in shape between the normal and logistic distributions, the value of β_2 [kurtosis] for logistic is 4.2, considerably different from the value ($\beta_2 = 3$) for the normal distribution". Therefore, it is of interest to have a test that distinguishes between these two distributions with good accuracy.

2. Saddlepoint Approximation

In this section, we derive the saddlepoint approximation to the distribution of the RML statistic for testing normal against logistic distribution (1.1) and for testing logistic against normal distribution (1.2). Then, for each case, we compare the saddlepoint approximation with the normal and Edgeworth approximations, and also with the exact values (based on 100,000 simulations), by presenting four graphs for (a) the density function, (b) tail probabilities, (c) relative error, and (d) power. (Details of the computational methods are available from the authors.) The results show that the saddlepoint approximation is more accurate than the normal and Edgeworth approximations. To derive the saddlepoint approximations, we follow the general procedure presented in Rasekhi and Sadooghi-Alvandi (2008, Sec. 3.2) for testing

$$f(x;\theta) = \frac{1}{\theta_1} f_0\left(\frac{x-\theta_0}{\theta_1}\right) \quad \text{vs.} \quad g(x;\lambda) = \frac{1}{\lambda_1} g_0\left(\frac{x-\lambda_0}{\lambda_1}\right),$$

where $f_0(x)$ and $g_0(x)$ are known density functions. Note that since the standard normal and logistic densities are symmetric, the procedure is simplified; in particular, $c^{(0)} = 0$ and $a_{rs} = a^*_{rs} = 0$, for $r \neq s$.

2.1. *Testing Normal versus Logistic*

Consider the problem of testing normal against logistic distribution (1.1), i.e., f and g denote the densities of normal and logistic distributions, respectively, Hence,

$$\hat{k}_{1n} = 0.00951, \quad \hat{k}_{2n} = 0.01368n^{-1},$$

(i.e., the normal approximation for the distribution of T_n^N is $N(0.00951, 0.01368\,n^{-1})$) and

$$\tilde{k}_{1n} = 0.00951 + 0.07954\,n^{-1}, \quad \tilde{k}_{2n} = 0.01368\,n^{-1} - 0.10345\,n^{-2} + 2.26253\,n^{-3},$$
$$\tilde{k}_{3n} = -0.00972\,n^{-2}, \quad \tilde{k}_{4n} = 0.01359\,n^{-3}.$$

Using these values, the saddlepoint approximations to the density function and distribution function of T_n^N are easily obtained; see Rasekhi and Sadooghi-Alvandi (2008, Sec. 3.2). The results are shown in Table 1 and Figs. 1(a–d), for sample size $n = 30$. These results show that saddlepoint approximation improves on the normal and Edgeworth approximations for the tail probabilities used in testing.

2.2. *Testing Logistic versus Normal*

Consider the problem of testing logistic versus normal (1.2), and let T_n^L denote the log of the RML. Note that $T_n^L = -T_n^N$, where T_n^N is defined by (2.2). Again, $c^{(0)} = 0$ (since the two densities are symmetric) and the value of $c^{(1)} = c$ is the solution of the equation $E[(Y/c)^2 - 1] = 0$, where Y has the standard logistic distribution. Solving this equation, we obtain $c = \pi/\sqrt{3}$. Also, $a_{rr} = E[A_r(Y)]$, where A_r is defined by (2.4). It is then easily verified that

$$a_{00} = -\frac{1}{3}, \quad a_{11} = -\frac{\pi^2}{9} - \frac{1}{3}, \quad a_{01} = a_{10} = 0,$$

Table 1

Comparison of saddlepoint (F_S), Edgeworth (F_E), and normal (F_N) approximations, for testing normal versus logistic (sample size $n = 30$). The values of t_α are the true critical points and the values in parentheses are relative errors (%) of the approximations

α	t_α	$F_S\ (t_\alpha)$	$F_E\ (t_\alpha)$	$F_N\ (t_\alpha)$
0.010	−0.050	0.013 (25.3)	0.019 (91.4)	0.003 (−72.6)
0.020	−0.039	0.026 (27.8)	0.033 (67.1)	0.011 (−44.4)
0.030	−0.033	0.039 (29.1)	0.042 (40.2)	0.024 (−21.3)
0.040	−0.028	0.051 (27.5)	0.049 (21.8)	0.038 (−5.45)
0.050	−0.025	0.063 (26.2)	0.055 (11)	0.054 (7.16)
0.060	−0.022	0.075 (24.6)	0.063 (4.62)	0.070 (16.6)
0.070	−0.020	0.086 (22.3)	0.070 (0.48)	0.086 (22.8)
0.080	−0.018	0.096 (20.5)	0.079 (−1.81)	0.102 (28)
0.090	−0.016	0.107 (19.4)	0.088 (−2.64)	0.120 (32.9)
0.100	−0.014	0.117 (17.4)	0.096 (−3.62)	0.136 (35.6)

Table 2
Comparison of saddlepoint (F_S), Edgeworth (F_E), and normal (F_N) approximations, for testing logistic versus normal (sample size $n = 30$). The values of t_α are the true critical points and the values in parentheses are relative errors (%) of the approximations

α	t_α	F_S (t_α)	F_E (t_α)	F_N (t_α)
0.010	−0.042	0.070 (596)	−0.420 (−4304)	0.082 (720)
0.020	−0.039	0.076 (277)	−0.384 (−2021)	0.094 (372)
0.030	−0.036	0.080 (167)	−0.343 (−1242)	0.104 (248)
0.040	−0.035	0.084 (109)	−0.305 (−863)	0.112 (180)
0.050	−0.033	0.087 (73)	−0.267 (−634)	0.119 (138)
0.060	−0.032	0.090 (49)	−0.228 (−479)	0.126 (110)
0.070	−0.031	0.092 (32)	−0.191 (−373)	0.132 (89)
0.080	−0.029	0.095 (18)	−0.155 (−294)	0.138 (73)
0.090	−0.028	0.097 (7.6)	−0.119 (−232)	0.144 (60)
0.100	−0.028	0.099 (−1.2)	−0.087 (−187)	0.149 (49)

(i.e., the normal approximation for the distribution of T_n^L is $N(0.01436, 0.04861\, n^{-1})$) and

$$\tilde{k}_{1n} = 0.01436 + 0.3\, n^{-1}, \quad \tilde{k}_{2n} = 0.04861\, n^{-1} - 1.44769\, n^{-2} + 27.22837\, n^{-3},$$
$$\tilde{k}_{3n} = 0.14124\, n^{-2}, \quad \tilde{k}_{4n} = 0.99661\, n^{-3}.$$

Using these values, the saddlepoint approximations to the density function and distribution function of T_n^L are easily obtained. The results are shown in Table 2 and Figs. 2(a–d), for sample size $n = 30$. In this case also, the saddlepoint approximation improves on the normal approximation for the tail probabilities, although the difference is small. Note that the Edgeworth approximation gives negative values for the tail probabilities used in testing (0.001–0.1). Therefore, we omitted Edgeworth approximation in Figs. 2(b–d). Note also that convergence to normality is slower than in the previous case, because of the relatively large values of third and fourth cumulants in this case.

3. Comparison with the Most Powerful Invariant Test

In this section, we compare the powers of RML and most powerful invariant (MPI) test statistics. It can be shown that for the general location-scale testing problem (2.1), a most powerful invariant test is based on the statistic $S_n = \log(q_f/q_g)$, where

$$q_f = \int_{-\infty}^{\infty}\int_0^{\infty} e^{\ell_f(\theta_0,\theta_1)}\frac{1}{\theta_1}d\theta_1\, d\theta_0 \quad \text{and} \quad q_g = \int_{-\infty}^{\infty}\int_0^{\infty} e^{\ell_g(\lambda_0,\lambda_1)}\frac{1}{\lambda_1}d\lambda_1\, d\lambda_0$$

(see Lehmann, 1986, Problem 5, p. 338). The hypothesis H_0 is rejected if $S_n < s_\alpha$, where s_α is the critical value for size α. However, as noted by Ducharme and Frichot (2003), the calculations are often intractable and "the MPI test has been

Table 3
Critical values (t_α and s_α) and powers of RML and MPI tests, for testing normal versus logistic (sample size $n = 30$)

α	t_α	s_α	Power (RML)	Power (MPI)
0.010	−0.049	0.214	0.087	0.087
0.020	−0.039	0.288	0.127	0.127
0.030	−0.033	0.343	0.159	0.159
0.040	−0.028	0.390	0.187	0.187
0.050	−0.025	0.427	0.210	0.211
0.060	−0.022	0.463	0.233	0.233
0.070	−0.020	0.495	0.253	0.254
0.080	−0.018	0.525	0.272	0.272
0.090	−0.016	0.553	0.290	0.290
0.100	−0.014	0.581	0.307	0.307

confined to a limited pairs of densities." (They proposed a "quasi MPI" test statistic, $\widehat{S}_n = \log(\hat{q}_f/\hat{q}_g)$, by deriving "quite accurate" approximations for q_f and q_g. However, they did not give the *general form* of the approximate distribution of $\widehat{S}_n$, but suggested that, for each value of n, the distribution may be found by simulation. They also noted that the dominant part of $\widehat{S}_n$ is nT_n, so their test behaves asymptotically as RML.)

We compared the power of MPI and RML for sample of size $n = 30$, based on 100,000 simulations. (For the MPI test, we calculated the integrals q_f and q_g numerically, rather than using the approximations proposed by Ducharme and

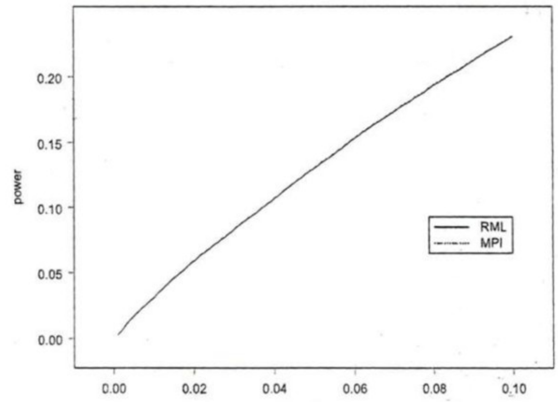

Figure 4. Comparison of the powers of RML and MPI tests, for testing logistic versus normal (sample size $n = 30$).

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